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# The description of $F_2$ at small $x$ incorporating angular ordering

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## Abstract

We study the perturbative QCD description of the HERA measurements of  $F_2(x, Q^2)$  using a gluon distribution that is obtained from an evolution incorporating angular ordering of the gluon emissions, and which embodies both GLAP and BFKL dynamics. We compare the predictions with recent HERA data for  $F_2$ . We present estimates of the charm component  $F_2^c(x, Q^2)$  and of  $F_L(x, Q^2)$ .

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Deep inelastic electron-proton scattering experiments at HERA have measured the structure function  $F_2(x, Q^2)$  in the previously unexplored small  $x$  regime,  $x \lesssim 10^{-3}$ . The values of  $F_2$  are found to rise rapidly with decreasing  $x$  [1, 2]. These measurements have stimulated much theoretical activity and the small  $x$  behaviour of  $F_2$  has been interpreted using perturbative QCD from several different viewpoints. The interpretation is complicated by the need to provide non-perturbative input.

In fact the present data for  $F_2$  can be well described by traditional Altarelli-Parisi (or GLAP) evolution in the next-to-leading order approximation. The data imply a steep gluon (that is a gluon density which increases as  $x$  decreases) even at low  $Q^2$  values. We may input the steep  $x$  behaviour directly into the starting distributions at some input scale, say  $Q_0^2 = 4 \text{ GeV}^2$  [3], or alternatively we may generate it from “non-singular” or “flat”  $x$  distributions at some low scale, such as  $Q_0^2 = 0.3 \text{ GeV}^2$  [4] or  $Q_0^2 = 1 \text{ GeV}^2$  [5], chosen so that the evolution length

$$\xi(Q_0^2, Q^2) = \int_{Q_0^2}^{Q^2} \bar{\alpha}_S(q^2) \frac{dq^2}{q^2}$$

is sufficiently long, where  $\bar{\alpha}_S \equiv 3\alpha_S/\pi$ . GLAP evolution amounts to the resummation of the leading (and next-to-leading)  $\log Q^2$  terms. At small  $x$  and large  $Q^2/Q_0^2$  it generates a steep double leading logarithmic (DLL) behaviour<sup>2</sup> of the form  $\exp(2[\xi(Q_0^2, Q^2) \log(1/x)]^{\frac{1}{2}})$ . Despite the apparent success of the GLAP approach, it is not the only way of generating a steep gluon compatible with the HERA data.

At sufficiently small  $x$  we must also resum the  $\alpha_S \log 1/x$  terms, unaccompanied by  $\log Q^2$ . This is accomplished by the BFKL equation. It generates a singular  $x^{-\lambda_L}$  behaviour of the unintegrated gluon distribution,  $f(x, k_T^2)$ , where  $\lambda_L = \bar{\alpha}_S 4 \ln 2$  for fixed  $\alpha_S$ . If a reasonable assumption is made to introduce the running of  $\alpha_S$  then the numerical solution of the BFKL equation again yields an  $x^{-\lambda}$  behaviour but with  $\lambda \simeq 0.5$  [6]. Using the  $k_T$ -factorization theorem the behaviour  $f(x, k_T^2) \sim x^{-\lambda}$  feeds through into  $F_2$  (and into  $F_L$ ). To be precise we have

$$F_i(x, Q^2) = \int \frac{dk_T^2}{k_T^2} \int_x^1 \frac{dx'}{x'} f(x', k_T^2) F_i^{\text{box}} \left( \frac{x}{x'}, k_T^2, Q^2 \right) + F_i^S \quad (1)$$

with  $i = 2, L$ , where  $F_i^S \simeq F_i(x, Q^2)$  at large  $x$ , but is a slowly varying function of  $x$  and  $Q^2$  at small  $x$ . The convolution in (1) is diagrammatically displayed in Fig. 1.  $F_i^{\text{box}}$  includes both the quark box and crossed box contributions which originate from virtual photon-virtual gluon  $q\bar{q}$  production. For the  $c\bar{c}$  component we take the quark mass to be  $m_c = 1.4$  (or 1.7) GeV. At sufficiently small  $x$  the  $x^{-\lambda}$  BFKL behaviour overrides the DLL form. To find precisely where this will happen requires a unified BFKL/GLAP formalism, as well as knowledge of the yet unknown next-to-leading  $\log 1/x$  contributions.

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<sup>2</sup>This form increases with decreasing  $x$  faster than any power of  $\log 1/x$  but slower than any power of  $1/x$ . A choice of singular starting distributions,  $xg, xq_{\text{sea}} \sim x^{-\lambda}$  with  $\lambda > 0$ , would therefore eventually override the DLL behaviour.

One approach which unifies the GLAP and BFKL formalisms is to recast the leading twist part of the BFKL  $k_T$ -factorization formula into collinear form in which the splitting and coefficient functions acquire the higher order  $\log 1/x$  contributions [7]. Several interesting phenomenological studies have developed from this formalism [8, 9, 10]. Actually the procedure is to use the BFKL equation for fixed  $\alpha_S$  to obtain  $\alpha_S/\omega$  power series expansions of the anomalous dimensions and coefficient functions, and then to let  $\alpha_S$  run. Here  $\omega$  is the moment index. In this way we retain the simplicity of GLAP evolution, but with splitting and coefficient functions which incorporate BFKL resummations<sup>3</sup>.

We can see that the basic quantity is the unintegrated gluon distribution  $f$ , which corresponds to the sum of (effective) gluon ladder diagrams. The unintegrated gluon satisfies the BFKL equation

$$f(x, k_T^2) = f^0(x, k_T^2) + \bar{\alpha}_S k_T^2 \int_x^1 \frac{dz}{z} \int \frac{dk_T'^2}{k_T'^2} \left[ \frac{f(z, k_T'^2) - f(z, k_T^2)}{|k_T'^2 - k_T^2|} + \frac{f(z, k_T^2)}{(4k_T'^4 + k_T^4)^{\frac{1}{2}}} \right], \quad (2)$$

where  $\bar{\alpha}_S \equiv 3\alpha_S/\pi$ . We notice the potential collinear singularity at  $k_T'^2 = 0$ . However, provided the driving term  $f^0$  is chosen to vanish at  $k_T^2 = 0$ , the structure of the equation guarantees that it is free of collinear singularities. This is a natural way<sup>4</sup> to regulate the singularity. It can be linked directly with the  $Q_0$  scheme advocated by Ciafaloni [11]. Since we stay in four dimensions we avoid factors which are characteristic of minimal subtraction (MS) schemes.

Here we work with the  $k_T$ -factorization formula, (1), and do not reduce the equation to collinear form. This allows us to study the effect of replacing the BFKL gluon with the gluon obtained from the CCFM equation [16]; a unified equation which embodies both the BFKL equation at small  $x$  and GLAP evolution at large  $x$ . The CCFM equation is based on the coherent radiation of gluons, which leads to an angular ordering of the emitted gluons.

Since the next-to-leading  $\log 1/x$  contributions are not yet known, the introduction of running  $\alpha_S$  into the BFKL equation<sup>5</sup> is, of necessity, subject to assumption. The most reasonable procedure is to take  $\alpha_S(k_T^2)$  in (2), so that the equation is compatible with the double-leading-logarithm limit of GLAP evolution. This prescription, however, generates a solution which differs from that obtained when the (fixed  $\alpha_S$ ) BFKL equation is first reduced to collinear form and then  $\alpha_S$  is allowed to run [14]. In other words the introduction of  $\alpha_S(k_T^2)$  in (2) gives a different solution to that obtained by allowing  $\alpha_S$  to run in the leading-twist collinear solution of the BFKL equation. Formally, however, the difference between the two approaches can be attributed to non-leading  $\ln 1/x$  effects. For instance in the region

$$\frac{\bar{\alpha}_S(Q^2)}{\omega} < \frac{\bar{\alpha}_S(Q_0^2)}{\omega} < \frac{1}{4 \ln 2}$$

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<sup>3</sup>There is freedom in assigning the BFKL resummations to the coefficient functions, splitting functions and the starting distributions. Various factorization schemes have been proposed [11, 12, 13].

<sup>4</sup>In fact the vanishing of the inhomogeneous term is ensured by the colour neutrality of the probed hadron.

<sup>5</sup>We thank M. Ciafaloni and L. Lipatov for valuable discussions on this point.

both methods give the same result up to these non-leading effects. However, the singularity structure in the moment or  $\omega$  plane is different in the two formulations. The collinear reduction of the BFKL equation, leading to conventional evolution from, say,  $Q_0^2$  to  $Q^2$ , contains a branch point singularity at  $\omega = \omega_L(Q_0^2) \equiv \bar{\alpha}_S(Q_0^2)4 \ln 2$  or, if this leading singularity is absorbed in the starting distributions, at  $\omega = \omega_L(Q^2)$ . On the other hand the solution with running  $\alpha_S$  directly incorporated into the BFKL equation does not contain the branch point singularity but rather it has (an infinite number of) poles in the  $\omega$  plane. The leading pole is well separated from the others with a position  $\omega_p \approx a\alpha_S(bk_0^2)$  where  $a$  and  $b$  are constants ( $b \approx 7$ ) and  $k_0$  delimits the infrared region. It turns out that  $\omega_p < \omega_L(k_0^2)$  [15].

Following ref. [16], we now implement angular ordering of the gluon emissions. The unintegrated gluon distribution is then a solution of the CCFM equation<sup>6</sup> [16] rather than the BFKL equation. In the small  $x$  region the CCFM equation may be approximated by

$$f(x, k_T^2, \bar{Q}^2) = f^0(x, k_T^2, \bar{Q}^2) + \bar{\alpha}_S k_T^2 \int_x^1 \frac{dz}{z} \int \frac{d^2 q}{\pi q^2} \Theta(\bar{Q} - zq) \Delta_R(z, q, k_T) \frac{1}{k_T'^2} f\left(\frac{x}{z}, k_T'^2, q^2\right) \quad (3)$$

with  $\mathbf{k}'_T = \mathbf{k}_T + \mathbf{q}$ , see eq. (18) of ref. [17].  $\Delta_R$  represents the virtual corrections which screen the  $1/z$  singularity and the theta function imposes the angular ordering on the real emissions. Eq. (11) of ref. [17] gives the explicit expression for  $\Delta_R$ . We note that the solution  $f$  depends on an additional scale  $\bar{Q}$  that is required to specify the maximum angle of gluon emission (which turns out to be essentially the scale  $\kappa$  of the probe, see Fig. 1). This equation has recently been solved numerically and the resulting gluon distribution has been compared with that obtained from the BFKL equation [17]. As anticipated, the angular ordering constraint suppresses the CCFM gluon at the lower  $\bar{Q}^2 = \kappa^2$  values. If we replace the angular-ordering constraint  $\Theta(\bar{Q} - zq)$  by  $\Theta(\bar{Q} - q)$  and set  $\Delta_R = 1$  then we obtain an equation which becomes equivalent to the Altarelli-Parisi (GLAP) equation in the double-leading-logarithm approximation (DLLA).

The small  $x$  approximation of the CCFM equation that we have used (see [17]) amounts to setting the Sudakov form factor  $\Delta_S = 1$  and to approximating the gluon-gluon splitting function by its singular term as  $z \rightarrow 0$ , that is  $P_{gg} \simeq 6/z$ .  $\Delta_S$  represents the virtual corrections which cancel the singularities at  $z = 1$ . We account for the remaining finite terms in  $P_{gg}$  by multiplying the solution  $f(x, k_T^2, \bar{Q}^2)$  by the factor

$$\exp\left(-A \int^{\bar{Q}^2} \bar{\alpha}_S(q^2) \frac{dq^2}{q^2}\right) \quad (4)$$

where  $A$  is defined by

$$\int_0^1 z^\omega P_{gg}(z) dz \simeq \frac{6}{\omega} - 6A. \quad (5)$$

That is  $A = (33 + 2n_f)/36$ , where the number of active flavours  $n_f = 4$ .

Fig. 2 compares the CCFM and DLLA predictions for  $F_2$  with the recent HERA measurements [1, 2]. The predictions are obtained by first determining the gluon distribution

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<sup>6</sup>The CCFM equation incorporates part of the non-leading  $\log 1/x$  contributions.

$f(x, k_T^2, \overline{Q}^2)$  by iteration of (3)<sup>7</sup> in the domain  $k_T^2 > k_0^2 = 1 \text{ GeV}^2$  starting from a “flat” driving term of the form  $3(1-x)^5 \exp(-k_T^2/k_a^2)$  with  $k_a^2 = 1 \text{ GeV}^2$ , that is exactly as in ref. [17]. We correct for the small  $x$  approximation by multiplying the gluon distribution  $f$  by the factor shown in (4) and then predict  $F_2$  from the  $k_T$ -factorization formula (1) with an infrared cut-off,  $k_T^2 > k_0^2$ . For  $F_2^S$  we use the value of  $F_2(x, Q^2)$  obtained from the MRS(A') set of partons [3] at  $x = 0.1$ , and extrapolate below 0.1 assuming the normal  $x^{-0.08}$  “soft” behaviour.

From Fig. 2 we see that the CCFM and DLLA predictions coincide at large  $x$ , as indeed they should. The two schemes start to differ at small  $x$  and Fig. 2 indicates the value of  $x$  at which the resummation effects become important. It is evident that once a background is added to the small  $x$  behaviour predicted by the CCFM equation then a good description of the HERA data is obtained. The prediction lies between the GRV and MRS(A') values. It should be noted that the CCFM calculation is not a fit to the HERA data, but simply a solution of the evolution equation incorporating angular ordering. The rise of the gluon, and hence of  $F_2$ , is generated by the evolution equation and hence is within the domain of perturbative QCD. Of course the perturbative QCD prediction is not absolute. The normalisation depends on the choice of  $k_0^2$ , which delimits the infrared region, and also on the choice of the driving term. Also the normalisation depends on the choice of the lower limit of integration in (4). Here we take this to be  $\overline{Q}_0^2 = 1 \text{ GeV}^2$ . Recall that the correction factor (4), and hence  $\overline{Q}_0$ , only occurs because we solve a simplified form of the CCFM equation appropriate to the small  $x$  region. In summary there is some freedom in the normalisation of  $F_2$ , though the prediction of the shape of the  $x$  dependence is characteristic of the CCFM equation. It is encouraging that the physically reasonable choice  $k_0^2 = \overline{Q}_0^2 = 1 \text{ GeV}^2$  gives such a satisfactory description of the HERA data.

For completeness we show in Figs. 3 and 4 respectively the predictions for the longitudinal structure function  $F_L(x, Q^2)$  and for the charm component of  $F_2$ , which we denote by  $F_2^c(x, Q^2)$ . In each case we specify the background or “soft” contribution  $F_i^S$  at  $x = 0.1$  to be given by the MRS(A') predictions, and extrapolate below 0.1 using the  $x^{-0.08}$  “soft” behaviour. For the charm component  $F_2^c$  the CCFM predictions for  $m_c = 1.4$  and  $1.7 \text{ GeV}$  are shown; the argument of the running coupling is taken to be  $\kappa^2 + m_c^2$  (where  $\kappa^2$  is shown in Fig. 1). The predictions of  $F_L$  and  $F_2^c$  obtained from GRV and MRS(A') partons are also shown in Figs. 3 and 4.

Since the CCFM values of  $F_2$  agree with the HERA data, we can regard the charm component  $F_2^c$  as an absolute prediction. The charm component of the MRS(A') partons has been fixed to be in agreement with the EMC measurements [18] of  $F_2^c$  which lie in the region  $x \sim 0.1$ . Indeed we see these data barely extend into the kinematic region shown in Fig. 4. It will be particularly informative to have measurements of  $F_2^c$  at HERA in the small  $x$  regime where resummation effects are expected to occur.

In summary we have shown that it is possible to obtain a good description of the HERA measurements of  $F_2$  from the solution of a unified evolution equation based on the angular

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<sup>7</sup>The DLLA prediction is obtained as defined above, that is by taking  $\Delta_R = 1$ , and  $z = 1$  in the  $\Theta$  function in (3).

ordering of the emitted gluons. The gluon distribution  $f(x, k_T^2, \overline{Q}^2)$  was obtained by iteration starting from a driving term of the form  $3(1-x)^5 \exp(-k_T^2/k_a^2)$ , and the structure function  $F_2$  was then determined via the  $k_T$ -factorization formula  $F_2 = f \otimes F_2^{\text{box}}$ . The steepness of the gluon, and of  $F_2$ , with decreasing  $x$ , is generated by the evolution equation. In this way we identified the regime where the  $\ln(1/x)$  resummations become important.

However, our treatment is only a first step. There are several reasons why it may overestimate the rise, particularly at low  $Q^2$ . First we have to find a realistic way to impose energy-momentum conservation of the emitted gluons. Second, we have ignored gluon shadowing corrections. These are expected to be small in the HERA regime, as evidenced by the persistent rise of the  $F_2$  data with decreasing  $x$  for  $Q^2$  as low as  $Q^2 = 2 \text{ GeV}^2$ . Last, but not least, the full next-to-leading  $\ln(1/x)$  contribution is unknown at present. This is needed to check the prescription for the running of  $\alpha_S$  and to specify the scale dependence.

Clearly the agreement of our CCFM predictions with the small  $x$  measurements of  $F_2$  do not imply angular ordering effects have been firmly established. GLAP and BFKL evolution can give an equally good description. There are two characteristic features of the gluon distribution  $f(x, k_T^2, \overline{Q}^2)$  obtained from an evolution equation which includes a resummation of  $\ln(1/x)$  terms. Namely a steep rise of  $f$  with decreasing  $x$  which is accompanied by a diffusion in  $\ln k_T^2$ .  $F_2$  measures only the rise. A distinctive test will involve both features. For this we need to explore final state processes such as deep inelastic events containing an identified energetic forward jet. Here we have focused on  $F_2$  and obtained predictions based on angular-ordered evolution which embodies both BFKL and GLAP resummations. Moreover, we have also presented values for the charm component  $F_2^c$  and the longitudinal structure function  $F_L$ . Valuable insight into the properties of the gluon can be obtained from the measurement of these structure functions at HERA.

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## Figure Captions

Fig. 1 Pictorial representation of the  $k_T$ -factorization formula, that is of the convolution  $F_i = f \otimes F_i^{\text{box}}$  contained in eq. (1) with  $i = 2, L$ .  $f(x', k_T^2)$  is the unintegrated gluon distribution and  $F_i^{\text{box}}$  is the off-shell gluon structure function, which at lowest order is determined by the quark box (and “crossed” box) contributions. The integration variables,  $x'$  and  $k_T^2$ , are respectively the longitudinal fraction of the proton’s momentum and the transverse momentum carried by the gluon which dissociates into the  $q\bar{q}$  pair.

Fig. 2 A comparison of the HERA measurements of  $F_2$  [1, 2] with the predictions obtained from the  $k_T$ -factorization formula (1) using for the unintegrated gluon distribution  $f$  the solutions of the CCFM equation (continuous curve), and the DLL-approximation (dot-dashed curve) of this equation. We also show the values of  $F_2$  obtained from collinear factorization using the MRS(A') [3] and GRV [4] partons.

Fig. 3 The continuous curve is the prediction for the longitudinal structure function of the proton,  $F_L(x, Q^2)$  obtained by solving the CCFM equation for the gluon and using the  $k_T$ -factorization formula (1). The values of  $F_L$  obtained from GRV [4] and MRS(A') [3] partons are also shown. The GRV prediction includes a charm component only at leading order.

Fig. 4 The continuous curves are the predictions for the charm component  $F_2^c$  of the proton structure function  $F_2$  obtained by solving the CCFM equation for the gluon and using the  $k_T$ -factorization formula with  $m_c = 1.4$  (upper) and  $1.7$  GeV (lower curve). The values of  $F_2^c$  obtained from GRV at leading order [4] and from MRS(A') [3] partons are also shown. The next-to-leading order GRV prediction lies below the leading order result; for example at  $Q^2 = 15$  GeV $^2$  and  $x = 10^{-4}$  it is shown by a small cross. Also shown are EMC data [18] at adjacent  $\langle Q^2 \rangle$  values, assuming that the  $c \rightarrow \mu + X$  branching ratio is 8%.

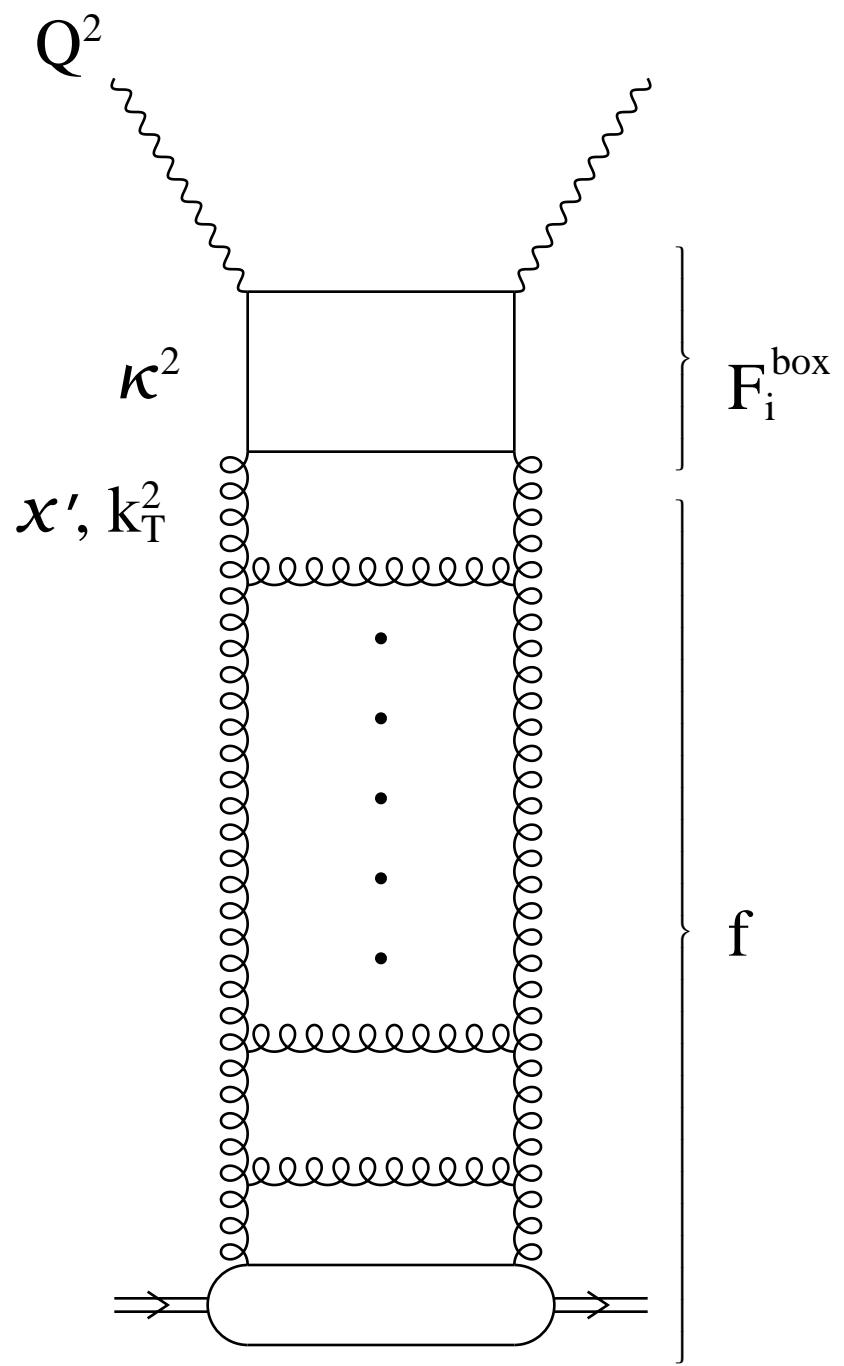
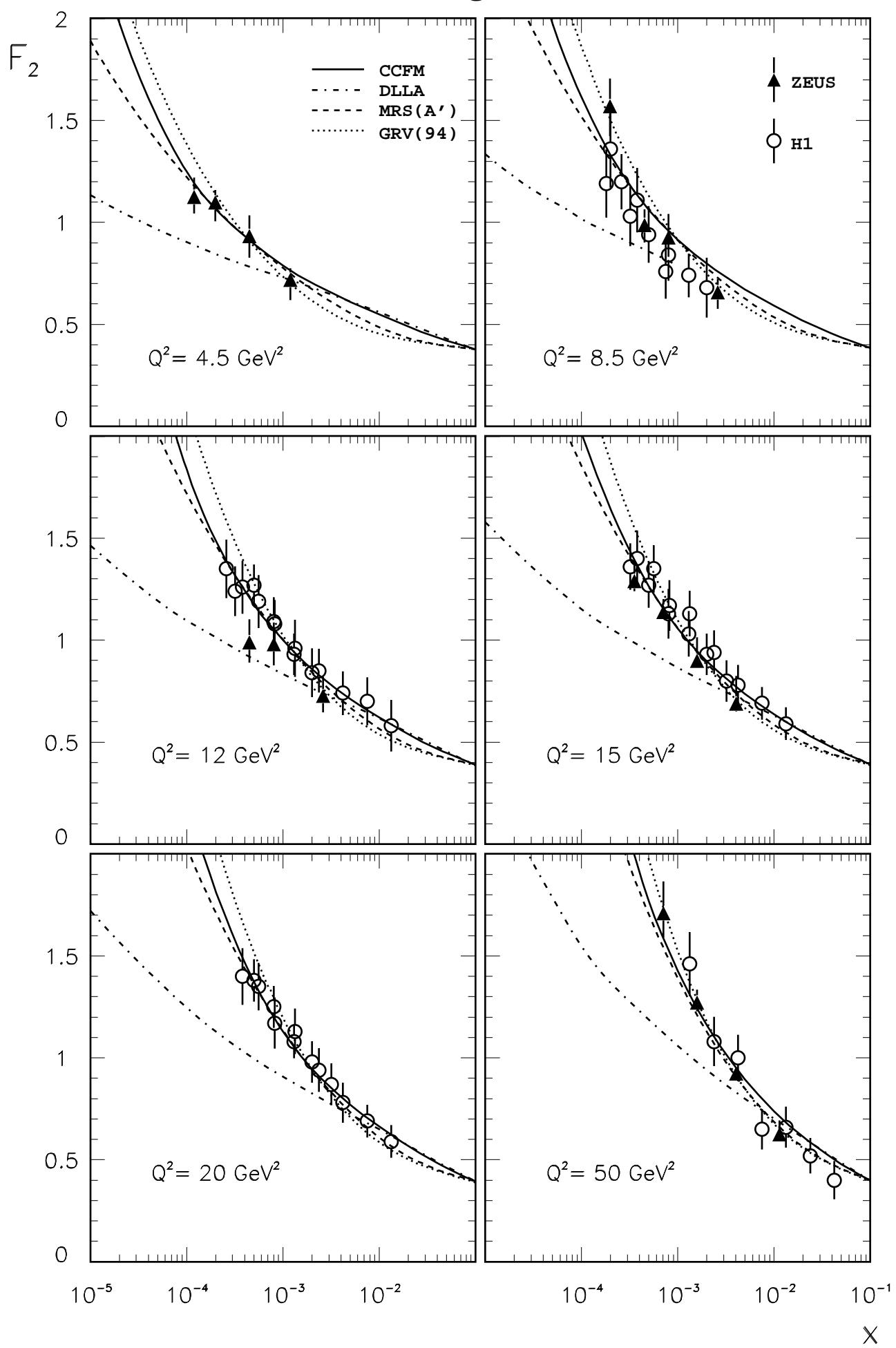
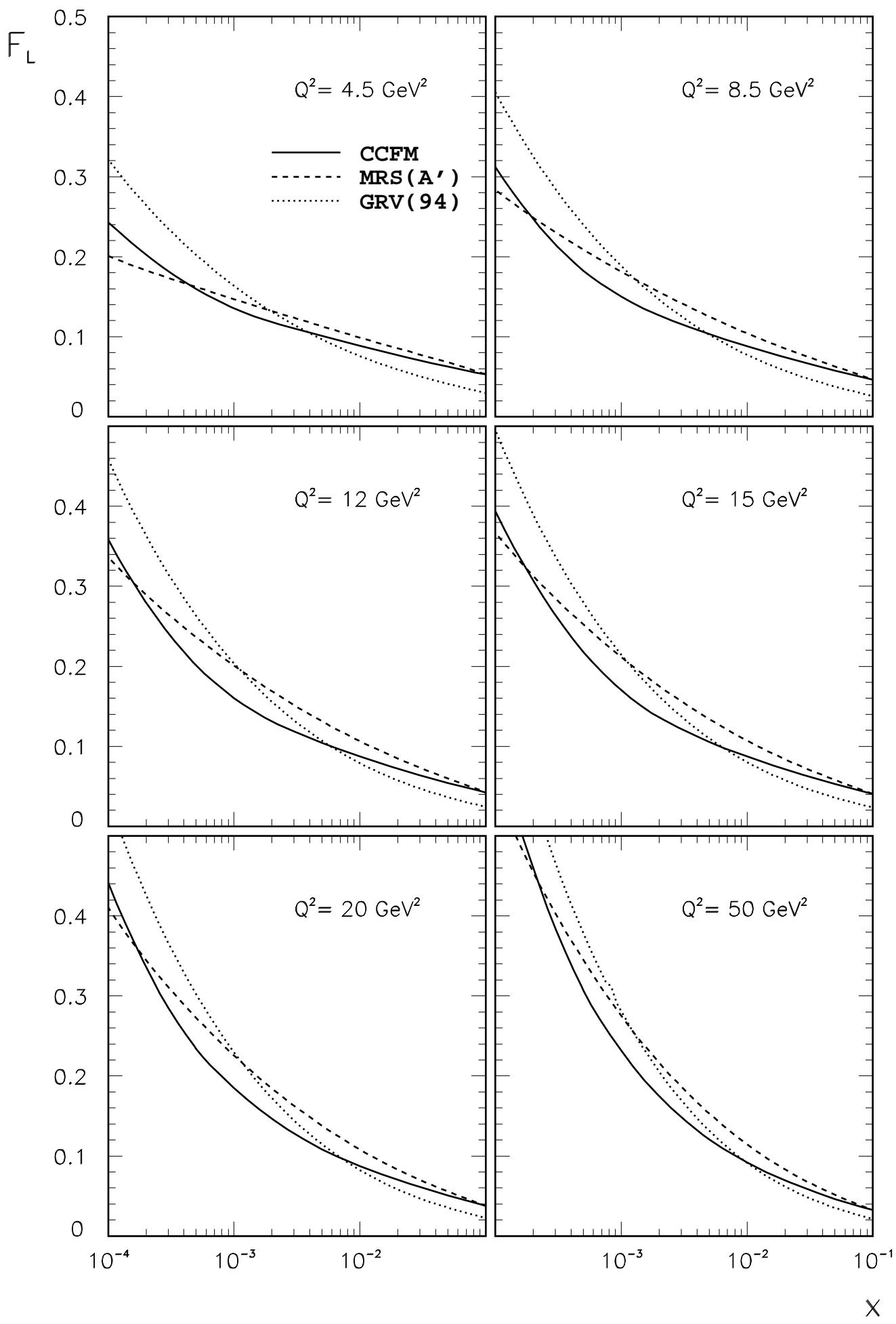


Fig. 1

# Fig. 2



**Fig. 3**



**Fig. 4**

